

Department of Electrical and Computer Systems Engineering

Technical Report MECSE-3-1996

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Problem Formulation

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12 April 1996

Technical Report: TR-96-3

ISBN: 0 7326 0661 6

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Abstract

Bilinear Matrix Inequality (BMI) is a potentially powerful tool for solving many robust control problems, however the feasibility of BMI is an \mathcal{NP} -hard problem [1]. In this paper, BMI is approached by some conic form optimization problems. It explores the subspace and the cone which are naturally related to the BMI problem, shows that the feasibility of the BMI problem is equivalent to some conic form optimization problems, which have zero optimal value, or a cone-LCP problem with a rank-1 solution. This research raises many open questions in optimization which are related to the BMI problem.

Key Words: Robust control, Bilinear Matrix Inequality (BMI), conic form optimization, cone-LCP.

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1 Introduction

The application of control theory to engineering systems stands to offer great benefits in terms of improved performance, reduced costs, and increased reliability. In the case of complex interactive engineering systems, however, there continues to exist a significant gap between control theory and practice.

Prior to the 1980's, the gap was attributable in large part to the fact that the applicability of the so-called *modern* control theory was limited to somewhat idealised plant models, where little if any account was taken of plant uncertainties and system nonlinearities. Over the past 15 years, control systems researchers have attempted to overcome these deficiencies particularly by extending the theory to include the effects of: (i) modelling uncertainties (the so-called robust control approach); and (ii) system nonlinearities. The former approach has evolved into a *post-modern* control theory wherein more realistic representations of plant uncertainties (for example parametric and non-parametric) can now be accommodated elegantly in the theoretical framework. It can be said that while the class of systems for which nonlinear control design techniques are applicable has been significantly expanded, it remains the fact that the applicability of nonlinear control design techniques to complex multivariable systems is still somewhat restricted. In this paper, attention is restricted to the post-modern robust control approach.

Central to this approach is the formulation of analysis and synthesis problems in terms of matrix inequalities. In particular, there are two categories of matrix inequalities now known to be of prime importance (see notations at the end of § 1):

Definition 1.1 Linear matrix inequality LMI [2]

Given parameters $F_i \in \mathcal{S}^p$ find any decision vector $\mathbf{x} \in \mathcal{R}^m$ such that:
 $\mathcal{A}(\mathbf{x}) := \sum_{i=1}^m x_i F_i \succ 0$ ($\mathcal{A}(\mathbf{x}) \in \mathcal{S}_{++}^p$).

Definition 1.2 Bilinear matrix inequality BMI [3]

Given parameters $F_{ij} \in \mathcal{S}^p$ find any pair of decision vectors $\mathbf{x} \in \mathcal{R}^m$ and $\mathbf{y} \in \mathcal{R}^n$ such that: $\mathcal{B}(\mathbf{x}, \mathbf{y}) := \sum_{j=1}^m \sum_{i=1}^n x_i y_j F_{ij} \succ 0$ ($\mathcal{B}(\mathbf{x}, \mathbf{y}) \in \mathcal{S}_{++}^p$).

LMIs are known to be useful for a number of control problems such as synthesizing a state-feedback controller to achieve a small-gain condition, computation of upper bounds to the structured singular value, and numerous other applications which are catalogued in [4]. In fact, for the small-gain state-feedback problem, they have proven to be more powerful and less restrictive than methods based on the solution of algebraic Riccati equations [5]. The convexity of LMI's ensures that LMI solutions may be computed in polynomial time using, for example, efficient interior point algorithms [2, 6].

It has recently been observed that a significantly wider class of robust control problems can be formulated in terms of the BMI [3]. Problems from this class include decentralised control, robust structured singular value synthesis and reduced-

order output feedback control, all of which cannot be formulated in terms of LMIs. Therefore, post-modern robust control *theory* has reached a certain maturity now, and essentially all that remains is to generate algorithms for the solution of BMI's. Unfortunately, BMI problems are non-convex (though they are bi-convex), and at present there are no systematic procedures for solving BMIs. Worst still, to check the solvability of BMI is \mathcal{NP} -hard [1]. Hence it is unlikely to find a polynomial-time algorithm for general BMI problems, and it can well be argued that the gap between theory and practice remains.

In this paper, we shall investigate the BMI problem by approaching it with some conic form optimization problems, which provides an insight into the cone specially related to BMI problems. These approaches raise new optimization problems and may also lead to some practical algorithms for solving the BMI problem in the future.

This paper is arranged as follows. In §2, the cones which are closely related to the BMI problem are studied, and a geometric interpretation is brought to the feasibility of the BMI problem. Some conic form optimization approaches to the BMI problem are discussed in §3, and the conclusions are summarized in §4.

The following notations and terminology are used throughout the paper (e.g. [10]):

- \mathcal{R}^n : the n dimensional Euclidean space
- \mathcal{R}_+^p : the nonnegative orthant of \mathcal{R}^p
- $\mathcal{R}^{p \times q}$: the set of all $p \times q$ matrices with real entries
- \mathcal{S}^p : the linear space of all symmetric $p \times p$ matrices
- \mathcal{S}_+^p : the cone of all $p \times p$ symmetric positive semidefinite matrices
- \mathcal{S}_{++}^p : the cone of all $p \times p$ symmetric positive definite matrices
- $Tr(\mathbf{A})$: the trace of a $p \times p$ matrix \mathbf{A}
- $\mathbf{A} \succeq 0$: \mathbf{A} is positive semidefinite
- $\mathbf{A} \succ 0$: \mathbf{A} is positive definite
- $\text{vec}\mathbf{A}$: the vector obtained by stacking up the columns of the matrix \mathbf{A}
- $\mathbf{A} \bullet \mathbf{H}$: $Tr(\mathbf{A}^\top \mathbf{H})$
- $\langle \cdot \rangle$: the inner product
- K^* : the dual cone of K , $K^* = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{k} \rangle \geq 0, \text{ for any } \mathbf{k} \in K\}$
- $\text{int}(K)$: interior point set of K
- $\|\mathbf{A}\|_F$: Frobenius norm of matrix \mathbf{A} .

2 BMI and related cones

A special case of BMI is the Linear Matrix Inequality (LMI) problem [2], which corresponds to $m = 1$ or $n = 1$ in BMI. For LMI, some polynomial time algorithms (e.g. [2]) have been developed, based on interior point methods. Extended Farkas' Lemma relates BMI with LMI.

Lemma 2.1 (*Extended Farkas' Lemma* [7, 8])

Given the symmetric matrices $\mathbf{A}_i \in \mathcal{S}^p$ ($1 \leq i \leq n$), the system $\sum_{i=1}^n x_i \mathbf{A}_i \succ \mathbf{0}$ has a solution if and only if the system

$$\sum_{i=1}^n (\mathbf{A}_i \bullet \mathbf{Z})^2 = 0 \quad (1)$$

has no non-zero solution in \mathcal{S}_+^p .

From Lemma 2.1, using the similar notations as in [8] (e.g. $\mathbf{F}_i^{\mathbf{y}}$, \mathcal{B} etc.) and the definition of the linear function M (see below), we have the following observation:

Remark 2.1 From Lemma 2.1, BMI has a solution

$$\iff \exists \mathbf{y} \in \mathcal{R}^n, \text{ such that for all } \mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq \mathbf{0},$$

$$\sum_{i=1}^m (\mathbf{F}_i^{\mathbf{y}} \bullet \mathbf{Z})^2 > 0 \quad (2)$$

Here $\mathbf{F}_i^{\mathbf{y}} = \sum_{j=1}^n y_j \mathbf{F}_{ij}$.

Therefore if $\mathbf{y} \in \mathcal{R}^n$ in (2) is found, then BMI is reduced to a LMI, i.e. find $\mathbf{x} \in \mathcal{R}^m$, such that

$$\sum_{i=1}^m x_i \mathbf{F}_i^{\mathbf{y}} \succ \mathbf{0} \quad (3)$$

For simplicity, without loss of generality, in the rest of this paper, we shall assume that $n = p$ (when $n \neq p$, the similar discussion can be carried out). Define a BMI related $p^2 \times p^2$ matrix \mathbf{F}_i as following

$$\mathbf{F}_i = [\mathbf{vec} \mathbf{F}_{i1}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{0}, \mathbf{vec} \mathbf{F}_{i2}, \mathbf{0}, \dots, \mathbf{0}, \dots, \mathbf{0}, \dots, \mathbf{vec} \mathbf{F}_{ip}] \quad (4)$$

and linear function

$$M(\mathbf{X}) = \sum_{i=1}^m \mathbf{F}_i \mathbf{X} \mathbf{F}_i^\top \quad (5)$$

Here matrices \mathbf{F}_{ij} ($i = 1, \dots, m; j = 1, \dots, p$) are from the original BMI problem (see Definition 1.2 in § 1). Then according to Remark 2.1, BMI has a solution

$$\iff \exists \mathbf{y} \in \mathcal{R}^n, \text{ such that for all } \mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq \mathbf{0},$$

$$(\mathbf{vec} \mathbf{Z})^\top M(\mathbf{X})(\mathbf{vec} \mathbf{Z}) > 0 \quad (6)$$

Here $\mathbf{X} = (\mathbf{vec} \mathbf{Y})(\mathbf{vec} \mathbf{Y})^\top$, and $\mathbf{Y} = \mathit{diag}(\mathbf{y})$. Or in other words, BMI has a solution

$$\iff \exists \mathbf{y} \in \mathcal{R}^n, \text{ such that for any } \mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq \mathbf{0},$$

$$((\mathbf{vec} \mathbf{Z})(\mathbf{vec} \mathbf{Z})^\top) \bullet M(\mathbf{X}) > 0 \quad (7)$$

Here $\mathbf{Y} = \mathit{diag}(\mathbf{y})$ and $\mathbf{X} = (\mathbf{vec} \mathbf{Y})(\mathbf{vec} \mathbf{Y})^\top$.

From (7), BMI is closely related to the set

$$\mathcal{B} = \{ \mathbf{A} \mid \mathbf{A} = \sum_{i=1}^t (\mathbf{vec} \mathbf{Z}_i)(\mathbf{vec} \mathbf{Z}_i)^\top, \mathbf{Z}_i \in \mathcal{S}_+^p, t \geq 1 \} \quad (8)$$

Indeed, from (7), BMI has a solution if and only if there exists an extreme point of \mathcal{B} , such that its image is an interior point of \mathcal{B}^* under the linear mapping M [8].

To investigate the set \mathcal{B} , different from [8], we shall restrict ourselves to the subspace \mathcal{L} (since if the discussion were carried out in the subspace $\mathcal{S}^{p^2 \times p^2}$ as in [8], $\text{int}(\mathcal{B})$ would be empty):

$$\mathcal{L} = \{ \mathbf{X} \in \mathcal{S}^{p^2 \times p^2} \mid \mathbf{X} \mathbf{e}_{i+(j-1)p} = \mathbf{X} \mathbf{e}_{j+(i-1)p}, (i = 1, \dots, p-1; i < j \leq p) \} \quad (9)$$

Here $\mathbf{X} \mathbf{e}_i$ is the i th columns of the matrix \mathbf{X} . From (9), we have $\mathcal{B} \subseteq \mathcal{L}$.

Some of the properties of \mathcal{B} are stated in the following theorem.

Theorem 2.1 *The dual of the set \mathcal{B} is*

$$\mathcal{C} = \{ \mathbf{A} \in \mathcal{L} \mid (\mathbf{vec} \mathbf{Z})^\top \mathbf{A} (\mathbf{vec} \mathbf{Z}) \geq 0, \mathbf{Z} \in \mathcal{S}_+^p \} \quad (10)$$

Both \mathcal{B} and \mathcal{C} are pointed closed convex cones.

Proof: $\mathcal{B}^* = \mathcal{C}$ is obvious. That both \mathcal{B} and \mathcal{C} are convex cones is also obvious.

(1) \mathcal{B} is pointed, i.e. $\mathcal{B} \cap (-\mathcal{B}) = \{\mathbf{O}\}$.
 $\forall \mathbf{X} \in \mathcal{B} \cap (-\mathcal{B}), \Rightarrow \mathbf{X} \in \mathcal{S}_+^{p^2}$ and $-\mathbf{X} \in \mathcal{S}_+^{p^2}$
 $\iff \forall \mathbf{y} \in \mathcal{R}^{p^2}, \mathbf{y}^\top \mathbf{X} \mathbf{y} = 0 \iff \mathbf{X} = \mathbf{O}$. i.e. $\mathcal{B} \cap (-\mathcal{B}) = \{\mathbf{O}\}$.

(2) \mathcal{C} is pointed. i.e. $\mathcal{C} \cap (-\mathcal{C}) = \{\mathbf{O}\}$.
 $\forall \mathbf{X} \in \mathcal{C} \cap (-\mathcal{C}),$ we have $\forall \mathbf{Z} \in \mathcal{R}_+^p,$

$$(\mathbf{vec} \mathbf{Z})^\top \mathbf{X} (\mathbf{vec} \mathbf{Z}) = 0 \quad (11)$$

Let $\mathbf{E}_{ij} \in \mathcal{S}^p$, whose entries are denoted by $e_{kl}^{(ij)}$ ($i, j = 1, \dots, p; k, l = 1, \dots, p$), and

$$e_{kl}^{(ij)} = \begin{cases} 1 & k = i, j \text{ and } l = i, j \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

Then for any $\mathbf{y} \in \mathcal{R}^p$, $\mathbf{y}^\top \mathbf{E}_{ij} \mathbf{y} = (y_i + y_j)^2 \geq 0$, which implies that $\mathbf{E}_{ij} \in \mathcal{S}_+^p$ ($i, j = 1, \dots, p$). Therefore it follows from (11) that

$$(\mathbf{vec} \mathbf{E}_{ij})^\top \mathbf{X} (\mathbf{vec} \mathbf{E}_{ij}) = 0 \quad (i, j = 1, \dots, p) \quad (13)$$

On the other hand, for any $i, j, k, l \in \{1, \dots, p\}$,

$$\mathbf{E}_{ij} + \mathbf{E}_{kl} \in \mathcal{S}_+^p \quad (14)$$

Thus

$$(\mathbf{vec}(\mathbf{E}_{ij} + \mathbf{E}_{kl}))^\top \mathbf{X}(\mathbf{vec}(\mathbf{E}_{ij} + \mathbf{E}_{kl})) = 0 \quad (15)$$

together with (13), we have

$$(\mathbf{vec}(\mathbf{E}_{ij}))^\top \mathbf{X}(\mathbf{vec}(\mathbf{E}_{kl})) = 0 \quad (i, j, k, l \in \{1, \dots, p\}) \quad (16)$$

Notice that

$$\mathcal{S}^p = \text{span}\{\mathbf{E}_{ij} \mid i, j = 1, \dots, p\} \quad (17)$$

we then have for all $\mathbf{W} \in \mathcal{S}^p$,

$$(\mathbf{vec}\mathbf{W})^\top \mathbf{X}(\mathbf{vec}\mathbf{W}) = 0 \quad (18)$$

Therefore $\mathbf{X} = \mathbf{O}$ since $\mathbf{X} \in \mathcal{L}$.

(3) \mathcal{C} is closed.

$\mathcal{C} = \mathcal{B}^* \Rightarrow \mathcal{C}$ is closed.

(4) \mathcal{B} is closed.

According to Carathéodory's Theorem [9], there exists an integer $N > 0$, ($N \leq \dim(\mathcal{S}^p) + 1$), such that the elements in \mathcal{B} can be expressed as a convex combination of N matrices of form $(\mathbf{vec}\mathbf{Z})(\mathbf{vec}\mathbf{Z})^\top$, $\mathbf{Z} \in \mathcal{S}_+^p$ (not necessarily distinct). Assume that the sequence $\{\mathbf{A}_k\} \subseteq \mathcal{B}$ converges to \mathbf{A} , with $\mathbf{A}_k = \sum_{i=1}^N \mathbf{Z}_i^{(k)} \mathbf{Z}_i^{(k)\top}$, $\mathbf{Z}_i^{(k)} \in \mathcal{S}_+^p$, then there exists an integer $k_0 > 0$, such that when $k \geq k_0$,

$$|A_k(i, j) - A(i, j)| \leq \frac{1}{2} \quad (i, j = 1, \dots, p^2) \quad (19)$$

Here $X(i, j)$ is the (i, j) th element of the matrix \mathbf{X} . Therefore there is an integer $\bar{k}_0 > 0$, and a scalar $M_A > 0$, such that when $k > \bar{k}_0$,

$$|Z_l^{(k)}(i, j)| \leq M_A \quad (l = 1, \dots, N; i, j = 1, \dots, p) \quad (20)$$

which implies that a subsequence of $\{\mathbf{Z}_i^{(k)}\} \subset \mathcal{S}_+^p$ ($i = 1, \dots, N$) has a limit $\mathbf{Z}_i^* \in \mathcal{S}_+^p$, since \mathcal{S}_+^p is closed. Thus

$$\mathbf{A} = \sum_{i=1}^N (\mathbf{vec}\mathbf{Z}_i^*)(\mathbf{vec}\mathbf{Z}_i^*)^\top \in \mathcal{B} \quad (21)$$

■

Remark 2.2 Since \mathcal{B} is a closed convex cone, we have

$$\mathcal{B} = \mathcal{B}^{**} = (\mathcal{B}^*)^* = \mathcal{C}^*$$

Theorem 2.2 *Both $\text{int}(\mathcal{B})$ and $\text{int}(\mathcal{C})$ are not empty, and*

$$\text{int}(\mathcal{C}) = \{\mathbf{A} \in \mathcal{L} \mid (\text{vec}\mathbf{Z})^\top \mathbf{A}(\text{vec}\mathbf{Z}) > 0, \forall \mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq 0\} \quad (22)$$

$$\text{int}(\mathcal{B}) \subseteq \{\mathbf{A} \in \mathcal{L} \mid (\text{vec}\mathbf{W})^\top \mathbf{A}(\text{vec}\mathbf{W}) > 0, \forall \mathbf{W} \in \mathcal{S}^p, \mathbf{W} \neq 0\} \quad (23)$$

Proof: According to [7], for any pointed closed convex cone K in a Hilbert space, $\text{int}(K)$ is not empty, and

$$\text{int}(K) = \{x \in K \mid \langle x, y \rangle > 0 \text{ for all } y \in K^*\} \quad (24)$$

Since $\mathcal{C}^* = \mathcal{B}$, (22) is obvious.

From (10), for any $\mathbf{W} \in \mathcal{S}^p$, $(\text{vec}\mathbf{W})(\text{vec}\mathbf{W})^\top \in \mathcal{C} = \mathcal{B}^*$. Hence if $\mathbf{A} \in \text{int}(\mathcal{B})$, $\mathbf{W} \in \mathcal{S}^p$, $\mathbf{W} \neq 0$, then we have

$$(\text{vec}\mathbf{W})(\text{vec}\mathbf{W})^\top \bullet \mathbf{A} = (\text{vec}\mathbf{W})^\top \mathbf{A}(\text{vec}\mathbf{W}) > 0 \quad (25)$$

Therefore (23) holds. ■

Remark 2.3

$$\text{int}(\mathcal{B}) \neq \{\mathbf{A} \in \mathcal{L} \mid (\text{vec}\mathbf{W})^\top \mathbf{A}(\text{vec}\mathbf{W}) > 0, \forall \mathbf{W} \in \mathcal{S}^p, \mathbf{W} \neq 0\} \quad (26)$$

To prove (26), we consider the cone $\hat{\mathcal{B}}$ which is defined as follows:

$$\hat{\mathcal{B}} = \{\mathbf{A} \in \mathcal{L} \mid \sum_{i=1}^t (\text{vec}\mathbf{W}_i)(\text{vec}\mathbf{W}_i)^\top, \mathbf{W}_i \in \mathcal{S}^p, t \geq 1\} \quad (27)$$

then we have

$$\hat{\mathcal{B}} = \mathcal{L} \cap \mathcal{S}_+^{p^2} \quad (28)$$

$$\hat{\mathcal{B}}^* = \hat{\mathcal{B}} \quad (29)$$

In fact, it is obvious that $\hat{\mathcal{B}} \subseteq \mathcal{L} \cap \mathcal{S}_+^{p^2}$. To prove $\mathcal{L} \cap \mathcal{S}_+^{p^2} \subseteq \hat{\mathcal{B}}$, consider the $1-1$ mapping $\phi: \mathcal{L} \rightarrow \mathcal{S}^q$ with $q = \frac{1}{2}p(p+1)$. Here ϕ is defined as: for any $\mathbf{X} \in \mathcal{L}$, $\phi(\mathbf{X})$ is obtained by deleting the $i + (j-1)p$ th ($j > i, i = 1, \dots, p-1$) columns and rows from \mathbf{X} . For any $\mathbf{X} \in \mathcal{L} \cap \mathcal{S}_+^{p^2}$, we have $\phi(\mathbf{X}) \in \mathcal{S}_+^q$, then there are $k \geq 0$, $\mathbf{u}_i \in \mathcal{R}^q$ ($i = 1, \dots, k$), such that $\phi(\mathbf{X}) = \sum_{i=1}^k \mathbf{u}_i \mathbf{u}_i^\top$ (e.g. [10]). Notice the special structure of the element in \mathcal{L} , there are $\mathbf{W}_i \in \mathcal{S}^p$, ($i = 1, \dots, k$), such that $\mathbf{X} = \sum_{i=1}^k (\text{vec}\mathbf{W}_i)(\text{vec}\mathbf{W}_i)^\top$, which implies that $\mathbf{X} \in \hat{\mathcal{B}}$. It follows from (28) that $\hat{\mathcal{B}} \subseteq \hat{\mathcal{B}}^*$. On the other hand, $\mathbf{X} \in \hat{\mathcal{B}}^* \iff \mathbf{X} \in \mathcal{L}$, and $\forall \mathbf{W} \in \mathcal{S}^p$, $(\text{vec}\mathbf{W})^\top \mathbf{X}(\text{vec}\mathbf{W}) \geq 0$. Thus $\phi(\mathbf{X}) \in \mathcal{S}_+^q$. Therefore, similar to the proof for (28), we have $\mathbf{X} \in \hat{\mathcal{B}}$, which implies that $\hat{\mathcal{B}}^* = \hat{\mathcal{B}}$.

By the same argument used in the proof of the Theorem 2.1, one can easily conclude that $\hat{\mathcal{B}}$ is a closed pointed cone, therefore it follows from (24) and (29) that

$$\text{int}(\hat{\mathcal{B}}) = \{\mathbf{A} \in \mathcal{L} \mid (\text{vec}\mathbf{W})^\top \mathbf{A} (\text{vec}\mathbf{W}) > 0, \forall \mathbf{W} \in \mathcal{S}^p, \mathbf{W} \neq \mathbf{0}\} \quad (30)$$

Now we are ready to prove (26) by showing the following example.
Let $p = 2$, and

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then $\mathbf{A} \in \text{int}(\hat{\mathcal{B}})$. However $\mathbf{A} \notin \mathcal{B}$.

\mathcal{B} is the cone which is naturally related to the BMI problem. However its structure is not clear. To some extent, such a situation shows the difficulty in developing practical algorithms for solving BMI at this stage.

3 BMI and Conic form optimization problems

In this section, we shall discuss that the BMI approached by conic form optimization problems (e.g. [6]). One property of the linear mapping defined in (4) is summarized in the following Lemma.

Lemma 3.1 *Let $\mathbf{X} = (\text{vec}\mathbf{Y})(\text{vec}\mathbf{Y})^\top$, $\mathbf{Y} \in \mathcal{S}_+^p$. Assume that $\mathbf{Y} = \mathbf{Q}^\top \mathbf{D} \mathbf{Q}$ is a spectral factorization of \mathbf{Y} . Here \mathbf{D} is a diagonal matrix with the eigenvalues of \mathbf{Y} on the diagonal, \mathbf{Q} is an orthogonal matrix of eigenvectors of \mathbf{Y} . Define $\mathbf{X}_1 = (\text{vec}\mathbf{D})(\text{vec}\mathbf{D})^\top$, then for all $\mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq \mathbf{0}$, we have*

$$(\text{vec}\mathbf{Z})^\top M(\mathbf{X})(\text{vec}\mathbf{Z}) > 0 \iff (\text{vec}\mathbf{Z})^\top M(\mathbf{X}_1)(\text{vec}\mathbf{Z}) > 0$$

Proof: First we note that

$$\text{vec}\mathbf{D} = \text{vec}(\mathbf{Q}\mathbf{Y}\mathbf{Q}^\top) = (\mathbf{Q} \otimes \mathbf{Q})\text{vec}\mathbf{Y} \quad (31)$$

Here \otimes is Kronecker product [10]. Thus for all $\mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq \mathbf{0}$

$$\begin{aligned} (\text{vec}\mathbf{Z})^\top M(\mathbf{X}_1)(\text{vec}\mathbf{Z}) &= (\text{vec}\mathbf{Z})^\top \left(\sum_{i=1}^m (\mathbf{F}_i \text{vec}\mathbf{D})(\mathbf{F}_i \text{vec}\mathbf{D})^\top \right) (\text{vec}\mathbf{Z}) \\ &= (\text{vec}\mathbf{D})^\top \left(\sum_{i=1}^m (\mathbf{F}_i \text{vec}\mathbf{Z})(\mathbf{F}_i \text{vec}\mathbf{Z})^\top \right) (\text{vec}\mathbf{D}) \\ &= ((\mathbf{Q} \otimes \mathbf{Q})\text{vec}\mathbf{Y})^\top \left(\sum_{i=1}^m (\mathbf{F}_i \text{vec}\mathbf{Z})(\mathbf{F}_i \text{vec}\mathbf{Z})^\top \right) ((\mathbf{Q} \otimes \mathbf{Q})\text{vec}\mathbf{Y}) \end{aligned}$$

$$\begin{aligned}
 &= (\mathbf{vec} \mathbf{Y})^\top (\mathbf{Q} \otimes \mathbf{Q})^\top \left(\sum_{i=1}^m (\mathbf{F}_i \mathbf{vec} \mathbf{Z})(\mathbf{F}_i \mathbf{vec} \mathbf{Z})^\top \right) ((\mathbf{Q} \otimes \mathbf{Q}) \mathbf{vec} \mathbf{Y}) \quad (32) \\
 &= (\mathbf{vec} \mathbf{Z})^\top (\mathbf{Q} \otimes \mathbf{Q})^\top \left(\sum_{i=1}^m (\mathbf{F}_i \mathbf{vec} \mathbf{Y})(\mathbf{F}_i \mathbf{vec} \mathbf{Y})^\top \right) ((\mathbf{Q} \otimes \mathbf{Q}) \mathbf{vec} \mathbf{Z}) \\
 &= \mathbf{vec}(\mathbf{Q} \mathbf{Z} \mathbf{Q}^\top)^\top M(\mathbf{X}) \mathbf{vec}(\mathbf{Q} \mathbf{Z} \mathbf{Q}^\top)
 \end{aligned}$$

Notice that $\mathbf{Z} \in \mathcal{S}_+^p$, $\mathbf{Z} \neq 0$, $\iff \mathbf{Q} \mathbf{Z} \mathbf{Q}^\top \in \mathcal{S}_+^p$, $\mathbf{Q} \mathbf{Z} \mathbf{Q}^\top \neq 0$. Thus it follows from (32) that $\forall \mathbf{Z} \in \mathcal{S}_+^p$, $\mathbf{Z} \neq 0$,

$$(\mathbf{vec} \mathbf{Z})^\top M(\mathbf{X})(\mathbf{vec} \mathbf{Z}) > 0 \iff (\mathbf{vec} \mathbf{Z})^\top M(\mathbf{X}_1)(\mathbf{vec} \mathbf{Z}) > 0$$

■

The following theorem shows that the BMI problem has a solution is equivalent to a conic form optimization problem has the optimal value zero.

Theorem 3.1 *BMI has a solution \iff*

for any $\alpha > 0$, problem (P_1) has the optimal value $v^ = 0$, where (P_1) is defined as*

$$\begin{aligned}
 &\min \quad \mathbf{I} \bullet \mathbf{X} - (\mathbf{vec} \mathbf{Z})^\top (\mathbf{vec} \mathbf{Z}) \\
 &s.t. \quad \mathbf{Z} \in \mathcal{S}_+^p \\
 (P_1) \quad &\begin{pmatrix} 1 & (\mathbf{vec} \mathbf{Z})^\top \\ \mathbf{vec} \mathbf{Z} & \mathbf{X} \end{pmatrix} \in \mathcal{S}_+^{p^2+1} \\
 &M(\mathbf{X}) - \alpha \mathbf{I}_{\mathcal{L}} \in \mathcal{B}^*
 \end{aligned} \quad (33)$$

Where $\mathbf{I}_{\mathcal{L}} \in \mathcal{L}$, $\mathbf{I}_{\mathcal{L}} = \phi^{-1}(\mathbf{I})$, \mathbf{I} is the identity matrix and ϕ is defined in Remark 2.3.

Proof: For any feasible point of (P_1) , the objective function value

$$v = \mathbf{I} \bullet \mathbf{X} - (\mathbf{vec} \mathbf{Z})^\top (\mathbf{vec} \mathbf{Z}) \geq 0 \quad (34)$$

Since

$$\begin{pmatrix} 1 & (\mathbf{vec} \mathbf{Z})^\top \\ (\mathbf{vec} \mathbf{Z}) & \mathbf{X} \end{pmatrix} \in \mathcal{S}_+^{p^2+1} \iff \mathbf{X} \succeq (\mathbf{vec} \mathbf{Z})(\mathbf{vec} \mathbf{Z})^\top \quad (35)$$

which implies that

$$\text{Tr}(\mathbf{X} - (\mathbf{vec} \mathbf{Z})(\mathbf{vec} \mathbf{Z})^\top) = \mathbf{I} \bullet \mathbf{X} - (\mathbf{vec} \mathbf{Z})^\top (\mathbf{vec} \mathbf{Z}) \geq 0$$

(\Rightarrow) BMI has a solution \Rightarrow there exist $\mathbf{0} \neq \mathbf{x} \in \mathcal{R}^m$,

$v \mathbf{0} \neq v \mathbf{y} \in \mathcal{R}^p$, such that

$\mathbf{X}^* = (\mathbf{vec} \mathbf{Z}^*)(\mathbf{vec} \mathbf{Z}^*)^\top$, $\mathbf{Z}^* = \text{diag}(\mathbf{y})$ satisfy $M(\mathbf{X}^*) \in \text{int}(\mathcal{B}^*)$. Without loss of generality, we assume that $\mathbf{y} \in \mathcal{R}_+^p$. Thus $\mathbf{Z}^* \in \mathcal{S}_+^p$. Let

$$\beta = \min_{\mathbf{Z} \in \mathcal{S}_+^p, \|\mathbf{Z}\|_F=1} (\mathbf{vec} \mathbf{Z})^\top M(\mathbf{X}^*)(\mathbf{vec} \mathbf{Z}) \quad (36)$$

we then have $\beta > 0$. Hence for any $0 < \bar{\alpha} < \beta$, $\forall \mathbf{Z} \in \mathcal{S}_+^p$, $\mathbf{Z} \neq 0$

$$(\mathbf{vec} \mathbf{Z})^\top (M(\mathbf{X}^*) - \bar{\alpha} \mathbf{I}_{\mathcal{L}})(\mathbf{vec} \mathbf{Z}) \geq (\beta - \bar{\alpha}) \|\mathbf{Z}\|_F^2 > 0 \quad (37)$$

Thus, for $0 < \bar{\alpha} < \beta$, (P_1) has the optimal solution $(\mathbf{X}^*, \mathbf{Z}^*)$ with the corresponding objective function value $v^* = 0$.

$\forall \alpha > 0$, define

$$\mathbf{X}_1^* = (\text{vec} \sqrt{\frac{\alpha}{\bar{\alpha}}} \mathbf{Z}^*) (\text{vec} \sqrt{\frac{\alpha}{\bar{\alpha}}} \mathbf{Z}^*)^\top = \frac{\alpha}{\bar{\alpha}} \mathbf{X}^* \quad (38)$$

then

$$M(\mathbf{X}_1^*) - \alpha \mathbf{I}_{\mathcal{L}} = \frac{\alpha}{\bar{\alpha}} \left(M\left(\frac{\bar{\alpha}}{\alpha} \mathbf{X}_1^*\right) - \bar{\alpha} \mathbf{I}_{\mathcal{L}} \right) = \frac{\alpha}{\bar{\alpha}} (M(\mathbf{X}^*) - \bar{\alpha} \mathbf{I}_{\mathcal{L}}) \in \mathcal{B}^* \quad (39)$$

Thus for any $\alpha > 0$, (P_1) has the optimal solution $(\mathbf{X}_1^*, \sqrt{\frac{\alpha}{\bar{\alpha}}} \mathbf{Z}^*)$ with the objective function value $v^* = 0$.

(\Leftarrow) For the given $\alpha > 0$, assume that (P_1) has the optimal solution $(\mathbf{X}^*, \mathbf{Z}^*)$ at which the objective function value $v^* = 0$, i.e.

$$\mathbf{I} \bullet \mathbf{X}^* - (\text{vec} \mathbf{Z}^*)^\top (\text{vec} \mathbf{Z}^*) = 0 \iff \mathbf{X}^* = (\text{vec} \mathbf{Z}^*) (\text{vec} \mathbf{Z}^*)^\top$$

since $\mathbf{X}^* - (\text{vec} \mathbf{Z}^*) (\text{vec} \mathbf{Z}^*)^\top \in \mathcal{S}_+^{p^2}$. Now let

$$\mathbf{Z}^* = \mathbf{Q}^\top \mathbf{Y} \mathbf{Q} \quad (40)$$

be a spectral factorization of \mathbf{Z}^* , where \mathbf{Y} is a diagonal matrix and $\mathbf{Q}^\top = \mathbf{Q}^{-1}$. Thus it follows from Lemma 3.1 and $M(\mathbf{X}^*) - \alpha \mathbf{I}_{\mathcal{L}} \in \mathcal{B}^*$ that

$$M(\mathbf{X}) - \alpha \mathbf{I}_{\mathcal{L}} \in \mathcal{B}^* \quad (41)$$

where $\mathbf{X} = (\text{vec} \mathbf{Y}) (\text{vec} \mathbf{Y})^\top$. Therefore BMI has a solution. ■

From Theorem 3.1, we have

Remark 3.1 If for a given $\alpha > 0$, (P_1) is not feasible, then BMI has no solution.

Remark 3.2 If for a given $\alpha > 0$, (P_1) has the global optimal value $v^* > 0$, then BMI has no solution.

Remark 3.3 Similar to Theorem 3.1, it can also be proved that

BMI has a solution \iff

for any $\alpha > 0$, the following problem (P_2) has the optimal value $v_2^* = 0$. Here (P_2) is defined as following:

$$(P_2) \quad \begin{array}{ll} \min & \mathbf{I} \bullet \mathbf{X} - \mathbf{y}^\top \mathbf{y} \\ \text{s.t.} & \mathbf{y} \in \mathcal{R}_+^p \\ & \begin{pmatrix} 1 & (\text{vec} \mathbf{Y})^\top \\ \text{vec} \mathbf{Y} & \mathbf{X} \end{pmatrix} \in \mathcal{S}_+^{p^2+1} \\ & M(\mathbf{X}) - \alpha \mathbf{I}_{\mathcal{L}} \in \mathcal{B}^* \end{array}$$

with $\mathbf{Y} = \text{diag}(\mathbf{y})$.

The conclusions in Remarks 3.1 and 3.2 are also applicable to problem (P_2) .

As noticed in §2, the cone \mathcal{B} has not been well studied, and its structure is not clear. Hence the feasibility problem of problems (P_1) and (P_2) might be difficult. Instead of dealing with (P_1) or (P_2) directly, we consider some relaxation to them as follows, where we take problem (P_1) as an example.

Given $\{\mathbf{Z}_i\} \in \mathcal{S}_+^p$, ($i = 1, \dots, r$; $r \geq \frac{1}{2}p(p+1)$). Define a polyhedral \mathcal{B}_r as

$$\mathcal{B}_r = \left\{ \sum_{i=1}^r \alpha_i (\text{vec } \mathbf{Z}_i)(\text{vec } \mathbf{Z}_i)^\top \mid \mathbf{Z}_i \in \mathcal{S}_+^p, \alpha_i \geq 0, i = 1, \dots, r \right\} \quad (42)$$

Choose $\{\mathbf{Z}_i\}$ ($i = 1, \dots, r$) in a way so that \mathcal{B}_r is full dimensional. Therefore \mathcal{B}_r is pointed and forms an inner approximation to the convex cone \mathcal{B} . In addition, \mathcal{B}_r^* is a polyhedral and $\mathcal{B}^* \subseteq \mathcal{B}_r^*$ since $\mathcal{B}_r \subseteq \mathcal{B}$. For any $\alpha > 0$, define a problem (P_{1r}) as follows:

$$(P_{1r}) \quad \begin{aligned} & \min \quad \mathbf{I} \bullet \mathbf{X} - (\text{vec } \mathbf{Z})^\top (\text{vec } \mathbf{Z}) \\ & \text{s.t.} \quad \mathbf{Z} \in \mathcal{S}_+^p \\ & \quad \begin{pmatrix} 1 & (\text{vec } \mathbf{Z})^\top \\ \text{vec } \mathbf{Z} & \mathbf{X} \end{pmatrix} \in \mathcal{S}_+^{p^2+1} \\ & \quad M(\mathbf{X}) - \alpha I_{\mathcal{L}} \in \mathcal{B}_r^* \end{aligned}$$

Then the feasible set of problem (P_1) is a subset of the feasible set of (P_{1r}) . Thus if the global optimal values of problems (P_1) and (P_{1r}) are denoted as v_1^* and v_{1r}^* respectively, we have $v_1^* \geq v_{1r}^*$. Hence both conclusions in Remarks 3.1 and 3.2 are also true for (P_{1r}) . If $v_{1r}^* = 0$, a new inner approximation \mathcal{B}_{r+1} to \mathcal{B} may need to be formed and then solve the corresponding problem $(P_{1(r+1)})$, an iterative procedure is then generated. How to update the inner approximation \mathcal{B}_r is worth considering.

We can also consider another relaxation to (P_1) as follows:

$$(\hat{P}_1) \quad \begin{aligned} & \min \quad \mathbf{I} \bullet \mathbf{X} - (\text{vec } \mathbf{Z})^\top (\text{vec } \mathbf{Z}) \\ & \text{s.t.} \quad \mathbf{Z} \in \mathcal{S}_+^p \\ & \quad \begin{pmatrix} 1 & (\text{vec } \mathbf{Z})^\top \\ \text{vec } \mathbf{Z} & \mathbf{X} \end{pmatrix} \in \mathcal{S}_+^{p^2+1} \\ & \quad M(\mathbf{X}) - \alpha I_{\mathcal{L}} \in \hat{\mathcal{B}}^* \end{aligned}$$

Then the feasible set of the problem (\hat{P}_1) is a subset of that of (P_1) . Denote the global optimal values for both problems (P_1) and (\hat{P}_1) as v_1^* and \hat{v}_1^* respectively, we have $v_1^* \leq \hat{v}_1^*$, which implies that BMI has a solution if $\hat{v}_1^* = 0$.

Similarly, we can define problem (P_{2r}) , (\hat{P}_2) and obtain the same conclusions as for (P_{1r}) and (\hat{P}_1) .

Conic form convex programming problems have been extensively studied, and some polynomial interior point methods have been developed (e.g. [6, 11, 12, 13, 14, 15, 16, 17, 18]). It is noticed that most available algorithms among them are for SDP

(semidefinite programming), and (P_1) , (P_2) , (P_{1r}) , (P_{2r}) , (\hat{P}_1) , (\hat{P}_2) are nonconvex optimization problems, therefore it may not be easy to get the global optimal solutions. However the feasible set for the problem (P_1) (or (P_2) , (P_{1r}) , (P_{2r}) , (\hat{P}_1) , (\hat{P}_2)) is a convex set. Hence new techniques for conic form, convex constrained optimization problem are required for solving the BMI problem.

Some strategies in nonlinear optimization might be applicable to this kind of problems. Take the problem (P_1) as an example. One can define the objective function differently, e.g. $\|\mathbf{X} - (\text{vec}\mathbf{Z})(\text{vec}\mathbf{Z})^\top\|_F^2$, approximate it locally (e.g. in a bounded ellipsoid) by a convex quadratic function, and then solve the corresponding (sub)problem, which is a conic form convex quadratic problem. However the global strategy is needed to be investigated before the following concept algorithm becomes practical.

Step 1. Set $k = 0$. Problem $(P^{(k)})$ is defined as (P_2) . The objective function is denoted as $v(\mathbf{X}_k, \mathbf{y}_k)$, and the corresponding feasible set as \mathcal{F}_k .

Step 2. Find a local minimum $(\mathbf{X}_k, \mathbf{y}_k)$ for $(P^{(k)})$. If $v(\mathbf{X}_k, \mathbf{y}_k) = 0$ stop, and the BMI has the solution $(\mathbf{X}_k, \mathbf{y}_k)$.

Step 3. Find a cutting plane $\mathcal{H}^k = \{(\mathbf{X}, \mathbf{y}) \mid \mathbf{A}_k \bullet \mathbf{X} + \mathbf{a}_k^\top \mathbf{y} = \mathbf{b}_k\}$. i.e. $(\mathbf{X}_k, \mathbf{y}_k) \in \mathcal{H}_-^k$, where $\mathcal{H}_-^k = \{(\mathbf{X}, \mathbf{y}) \mid \mathbf{A}_k \bullet \mathbf{X} + \mathbf{a}_k^\top \mathbf{y} < \mathbf{b}_k\}$, and if $(\mathbf{X}, \mathbf{y}) \in \mathcal{H}_-^k \cap \mathcal{F}_k$, $v(\mathbf{X}, \mathbf{y}) \geq v(\mathbf{X}_k, \mathbf{y}_k)$.

Step 4. Let $\mathcal{F}_{k+1} = \mathcal{H}_+^k \cap \mathcal{F}_k$, where $\mathcal{H}_+^k = \{(\mathbf{X}, \mathbf{y}) \mid \mathbf{A}_k \bullet \mathbf{X} + \mathbf{a}_k^\top \mathbf{y} \geq \mathbf{b}_k\}$. If $\mathcal{F}_{k+1} = \emptyset$, stop, and the BMI has no solution.

Step 5. set $k = k + 1$, go to Step 2.

Both Step 2 and Step 3 form the heart of the above algorithm. They stay as open questions at this stage, and each of them leads to an important research direction. They will also result in a practical algorithm for solving the BMI problem.

The following theorem provides another conic form optimization approach to BMI.

Theorem 3.2 *BMI has a solution \iff*

$\exists \mathbf{Q} \in \text{int}(-\mathcal{C})$, such that there is a rank-1 element in the feasible set of the cone-LP problem (P_3) . Here (P_3) is defined as

$$(P_3) \quad \begin{array}{ll} \min & \mathbf{C} \bullet \mathbf{X} \\ \text{s.t.} & \mathbf{X} \in \hat{\mathcal{B}} \\ & M(\mathbf{X}) + \mathbf{Q} \in \hat{\mathcal{B}}^* \end{array}$$

Here $\mathbf{C} \in \mathcal{R}^{p^2 \times p^2}$ is a constant matrix.

Proof: (\Rightarrow) BMI has a solution $\Rightarrow \exists \mathbf{Y} = \text{diag}(\mathbf{y})$, $\mathbf{y} \in \mathcal{R}^p$, such that $\mathbf{X} = (\text{vec} \mathbf{Y})(\text{vec} \mathbf{Y})^*$ satisfies that $M(\mathbf{X}) \in \text{int}(\mathcal{B}^*)$. Therefore there is a $\beta > 0$, such that

$$\min_{\mathbf{Z} \in \mathcal{S}_+^p, \|\mathbf{Z}\|_F=1} (\text{vec} \mathbf{Z})^\top M(\mathbf{X})(\text{vec} \mathbf{Z}) \geq \beta \quad (43)$$

On the other hand, for the above \mathbf{X} , there is a $\mathbf{W} \in \hat{\mathcal{B}}$ such that $\|\mathbf{W}\|_F = 1$, $\text{rank}(\mathbf{W}) = q - 1$, ($q = \frac{1}{2}p(p+1)$) and $\mathbf{X} \bullet \mathbf{W} = 0$.

Denote $\mathbf{Q}_\alpha = \alpha \mathbf{W} - M(\mathbf{X})$. Then for $\mathbf{Z} \in \mathcal{S}_+^p, \|\mathbf{Z}\|_F = 1$, we have

$$(\text{vec} \mathbf{Z})^\top \mathbf{Q}_\alpha (\text{vec} \mathbf{Z}) = \alpha (\text{vec} \mathbf{Z})^\top \mathbf{W} (\text{vec} \mathbf{Z}) - (\text{vec} \mathbf{Z})^\top M(\mathbf{X})(\text{vec} \mathbf{Z}) \leq \alpha - \beta \quad (44)$$

Thus if $\alpha > \beta$, we have for all $\mathbf{Z} \in \mathcal{S}_+^p, \|\mathbf{Z}\|_F = 1$,

$$(\text{vec} \mathbf{Z})^\top \mathbf{Q}_\alpha (\text{vec} \mathbf{Z}) < 0 \quad (45)$$

which implies that for all $\mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq 0$,

$$(\text{vec} \mathbf{Z})^\top \mathbf{Q}_\alpha (\text{vec} \mathbf{Z}) < 0 \quad (46)$$

i.e. $\mathbf{Q}_\alpha \in \text{int}(-\mathcal{C})$, and

$$M(\mathbf{X}) + \mathbf{Q}_\alpha = \alpha \mathbf{W} \in \hat{\mathcal{B}} = \hat{\mathcal{B}}^* \quad (47)$$

Thus for the above \mathbf{Q}_α , (P_3) has a rank-1 feasible point \mathbf{X} .

(\Leftarrow) $\exists \mathbf{X}_1 \in \hat{\mathcal{B}}, \text{rank}(\mathbf{X}_1) = 1$, such that $M(\mathbf{X}_1) + \mathbf{Q} \in \hat{\mathcal{B}}^* = \hat{\mathcal{B}}$ with $\mathbf{Q} \in \text{int}(-\mathcal{C})$. Therefore we assume that

$$\mathbf{X}_1 = (\text{vec} \mathbf{W})(\text{vec} \mathbf{W})^\top \quad (48)$$

with $\mathbf{W} \in \mathcal{S}^p$, then for all $\mathbf{W}_i \in \mathcal{S}^p$

$$(\text{vec} \mathbf{W}_i)^\top (M(\mathbf{X}_1) + \mathbf{Q})(\text{vec} \mathbf{W}_i) \geq 0 \quad (49)$$

$\Rightarrow \forall \mathbf{Z} \in \mathcal{S}_+^p, \mathbf{Z} \neq 0$

$$(\text{vec} \mathbf{Z})^\top M(\mathbf{X}_1)(\text{vec} \mathbf{Z}) \geq (\text{vec} \mathbf{Z})^\top (-\mathbf{Q})(\text{vec} \mathbf{Z}) > 0 \quad (50)$$

i.e. $M(\mathbf{X}_1) \in \text{int}(\mathcal{B}^*)$ and therefore BMI has a solution. ■

Remark 3.4 From the proof of Theorem 3.2, BMI has a solution if and only if there exists a $\mathbf{Q} \in \text{int}(-\mathcal{B}^*)$ and a rank-1 matrix $\mathbf{X} \in \hat{\mathcal{B}}$, such that

$$M(\mathbf{X}) + \mathbf{Q} \in \hat{\mathcal{B}}, \quad \mathbf{X} \bullet (M(\mathbf{X}) + \mathbf{Q}) = 0$$

Notice that $\hat{\mathcal{B}} \subseteq \mathcal{S}_+^{p^2}$, we have [19]

$$\mathbf{X} \bullet (M(\mathbf{X}) + \mathbf{Q}) = 0 \iff \mathbf{X}(M(\mathbf{X}) + \mathbf{Q}) = \mathbf{O}$$

Thus BMI has a solution if and only if there exists a $\mathbf{Q} \in \text{int}(-\mathcal{B}^*)$ such that the cone-LCP (Linear Complementarity Problem) problem (e.g. [22]):

$$\begin{aligned} \mathbf{X} &\in \hat{\mathcal{B}} \\ (M(\mathbf{X}) + \mathbf{Q}) &\in \hat{\mathcal{B}}^* \\ \mathbf{X}(M(\mathbf{X}) + \mathbf{Q}) &= \mathbf{O} \end{aligned}$$

has a rank-1 solution.

There are numerous literatures on cone-LCP (e.g. [20, 21, 22, 23, 24, 25, 26, 27]) and related problems. If the matrix \mathbf{Q} in Remark 3.4 is known, then BMI is reduced to a cone-LCP problem. Unfortunately, for the general BMI problem, to find such a \mathbf{Q} is not trivial, though it might be easier for some special cases.

4 Conclusions

In this paper, we reformularized the BMI problem by studying the relationship between the BMI problem and some conic form optimization problems. It brings the following additions to the research on BMI:

1. BMI is naturally related to the subspace \mathcal{L} and the closed convex cone $\mathcal{B} \in \mathcal{L}$. It is worth investigating the structure of the cone \mathcal{B} for solving the BMI problem.
2. BMI has a solution \iff the conic form optimization problems (P_1) ((P_2)) has the optimal value zero. Thus the feasibility of the BMI problem can be determined by solving (P_1) ((P_2)), which can also be explored, to some extent, by the feasibility and the solutions of the relaxation problems (P_{1r}) , (P_{2r}) , (\hat{P}_1) , (\hat{P}_2) . Therefore the research on conic form, convex constrained, nonconvex optimization is required.
3. BMI has a solution \iff the cone-LCP problem (P_3) has a rank-1 solution. The available algorithms can be applied to this cone-LCP problem provided the parameter \mathbf{Q} in (P_3) is known.

This paper shows where the difficulties are for designing practical algorithms for BMI problems at this stage, raises new requirements on optimization development, and also points out a few directions for further research related to BMI.

Acknowledgments

We sincerely thank Professor Masakazu Kojima, who showed interest in this work and patiently answered our questions. We appreciate the discussion with Dr. M. Shida and Dr. M. Mesbahi.

This work is heavily motivated by [8]. We thank Dr. M. Mesbahi kindly offered us this report [8].

Additionally, Dr. F. Crusca is indebted to Dr M. Safonov for guidance received on BMI research during a research visit to University of South California, Department of EE-Systems in the 1994 Fall Semester. He would additionally like to thank Dr G. P. Papavassilopoulos, Dr K. C. Goh, Dr S. Boyd, Dr A. Fradkov, Dr B. D. Craven and Dr L. Turan for various BMI related discussions.

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